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# EFFECTIVE MODELS OF COMPOSITE PERIODIC PLATES—II. SIMPLIFICATIONS DUE TO SYMMETRIES

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**Abstract**—This paper is aimed at investigating how material and geometrical symmetries of a composite plate simplify the formulae derived in Part I by the asymptotic method. It occurs that the symmetry with respect to the middle plane results in splitting the subsequent overall and local problems into membrane and bending problems. Additional assumptions of orthotropy of the material and of certain symmetries of the periodicity cells imply far-reaching simplifications, e.g. the vanishing of some terms in the first-order correctors for displacements, and the cancelation of discrepancies in the formulation of some boundary conditions. In the last section, a computational algorithm for evaluating the effective stiffnesses is suggested.

## 1. INTRODUCTION

The asymptotic solution of the elastostatic problem of a plate with periodic composite structure has been dealt with in Part I of this paper. Following the algorithm of Caillerie (1984) and making use of the results of the works by Kohn and Vogelius (1984) and Kalamkarov *et al.* (1987), we find the subsequent third term of the asymptotic expansion. In the general case considered in Part I, both the main ($P_{\text{hom}}$) as well as the subsidiary ($P'_{\text{hom}}$) homogenized problems are coupled. In the practically important case of a plate symmetric with respect to its middle plane, the membrane and bending deformations become independent. One can conjecture that such a symmetry property would result in essential reductions and decouplings of the overall as well as the auxiliary local problems. The consequences of this type of symmetry are addressed in Section 2 of the present paper.

The next section is aimed at examining the consequences of the orthotropy of the material and of the symmetries of the periodicity cell, with respect to its transverse central cross-sections. As known from the classical homogenization theory (Bourgat and Dervieux, 1978, Section 3, Theorem 7), by virtue of similar symmetry properties, the corrector term for displacements depends directly on the solutions of the homogenized problem and of the first-order basic cell problem, provided that the solution to the latter problem is subject to the normalizing condition (i.e. provided its average—over the rescaled cell of periodicity—vanishes). This simplification is used to evaluate the displacements within the framework of the homogenization theory. In the case of a cell with general properties, the corrector term for displacements involves an extra global function which is difficult to find. However, the adduced theorem by Bourgat and Dervieux (1978) is not exact since it is based on the additional assumption that this extra function vanishes on the boundary, which is in conflict with the requirement that the whole corrector term should vanish along the boundary. Thus, according to this approach the corrector term does not fulfil the boundary condition. However, this method seems to be the only rational approach since in the exact formulation of the boundary value problem for the corrector, the boundary condition becomes oscillatory, which would negate the idea of homogenization (cf. Bensoussan *et al.*, 1978). Similar, yet not identical points are at issue in Section 3.

In Section 4, we show that the primal homogenization problem for the initial plate (before scaling) can be arrived at directly by imposing displacement assumptions on the governing variational equation, defined on the product domain $\Omega \times \mathcal{Y}$, cf. Section 4 of Part I. This approach can be referred to as variationally-asymptotic since it cannot be carried out without introducing a small parameter and discarding the terms of higher order with respect to it. A purely variational derivation does not seem to exist.

In Section 5 we refer to the paper by Kohn and Vogelius (1984) and clear up the discrepancies between their results and the formulae derived in Part I. The last section is concerned with a computational algorithm for evaluating effective stiffnesses which utilizes the Galerkin approximation of the basic cell problems.

The denotations of the boundary value problems formulated in Part I [e.g. $(P^1_{\text{loc}})$] are preserved and the summation and notation conventions are also retained. In particular, the parentheses $\{\cdot\}$ and $\langle\cdot\rangle$ imply averaging over $Y$ and $\mathscr{Y}$, respectively; $Y$ and $\mathscr{Y}$ being defined in Part I. These brackets will not be used for any other meaning.

## 2. CONSEQUENCES OF SYMMETRY WITH RESPECT TO THE MIDDLE PLANE

The subject of consideration is deformations of the periodic plate, as described in Section 2, Part I. Additionally, we assume here that the geometry and properties of the plate are symmetric with respect to the plane $x_3 = 0$, viz. $C^{ijkl}_Z(x, x_3) = C^{ijkl}_Z(x, -x_3)$ and $x_3^+(x) = -x_3^-(x)$. Moreover, we assume that the planes $x_3 = \text{const}$ are planes of material symmetry, i.e. $C^{3\alpha\beta\gamma}_Z = C^{333\alpha}_Z = 0$.

As in Lewiński (1991) the problem $(P)$ is replaced by a family of problems $(P_\varepsilon)$ and then by $(\mathscr{P}_\varepsilon)$. In accordance with the assumptions of symmetry, we have here

$$C^{ijkl}\left(\frac{x_3}{\varepsilon}, \frac{x}{\varepsilon}\right) = C^{ijkl}\left(-\frac{x_3}{\varepsilon}, \frac{x}{\varepsilon}\right), \quad C^{3\alpha\beta\gamma} = C^{333\alpha} = 0$$

and

$$c^+(x/\varepsilon) = -c^-(x/\varepsilon) = c(x/\varepsilon),$$

$$G_+(x/\varepsilon) = G_-(x/\varepsilon) = G(x/\varepsilon), \quad \hat{y}_3 = y_3.$$

It is expedient to decompose the surface loadings into in-plane and anti-plane loadings, i.e.

$$\begin{aligned} r_\alpha^\pm &= S_\alpha(x, x/\varepsilon) \pm M_\alpha(x, x/\varepsilon)/2c(x/\varepsilon), \\ q^\pm &= S(x, x/\varepsilon) \pm M(x, x/\varepsilon). \end{aligned} \tag{1}$$

Let $S$ and $A$ represent the classes of even and odd functions in $y_3$, respectively. Under the assumptions considered, the solutions of the basic cell problems $(P^\alpha_{\text{loc}})$ possess the properties: $\Theta^{(\alpha\beta)}_\sigma, \Xi^{(\alpha\beta)}_3 \in S$; $\Theta^{(\alpha\beta)}_3, \Xi^{(\alpha\beta)}_\sigma \in A$. Hence, $A_0^{\gamma\delta\alpha\beta} \in S$, $E_0^{\gamma\delta\alpha\beta} \in A$ and as a consequence, the tensors $\mathbf{E}_z$, $\mathbf{F}_z$, which couple the homogenized constitutive relations, vanish. Similarly, one can conclude that the solutions to the local $(P^3_{\text{loc}})$ and $(P^4_{\text{loc}})$ problems are such that $\Lambda^{(\sigma\gamma\beta)}_\alpha$, $\Pi^{(\beta\gamma\delta)}_3 \in S$; $\Lambda^{(\sigma\gamma\beta)}_3$, $\Pi^{(\beta\gamma\delta)}_\alpha \in A$, which yield $\mathbf{B}_{z1} = \mathbf{0}$, $\mathbf{D}_{z2} = \mathbf{0}$. Thus, the $(P_{\text{hom}})$ problem splits into 2 problems, as follows. The in-plane problem: find $\mathbf{v} = (v_\alpha) \in [H^1_0(\Omega)]^2$ such that for every $\mathbf{w} \in [H^1_0(\Omega)]^2$

$$(P^S_{\text{hom}}) \qquad t\int_\Omega A_z^{\alpha\beta\lambda\mu} v_{\lambda,\mu} w_{\alpha,\beta}\, \mathrm{d}x = \int_\Omega s_\alpha w_\alpha\, x, \tag{2}$$

where $t = |\mathscr{Y}|/|Y|$ and

$$\langle\sigma_1^{\alpha\beta}\rangle = A_z^{\alpha\beta\lambda\mu} v_{\lambda,\mu}, \quad s_\alpha = 2\cdot\{S_\alpha(x, y)\cdot(G(y))^{1/2}\}; \tag{3}$$

and the bending problem: find $w \in H^2_0(\Omega)$ such that for every $\boldsymbol{\varphi} \in [H^1_0(\Omega)]^2$ and $v \in H^1_0(\Omega)$ we have

$$(P_{\text{hom}}^{A}) \quad t\int_{\Omega}[\langle y_3\sigma_1^{\alpha\beta}\rangle\varphi_{\alpha,\beta}+\langle\sigma_2^{\alpha 3}\rangle\varphi_\alpha]\,\mathrm{d}x = \int_{\Omega} m_\alpha\varphi_\alpha\,\mathrm{d}x, \quad t\cdot\int_{\Omega}\langle\sigma_2^{\alpha 3}\rangle v_{,\alpha}\,\mathrm{d}x = \int_{\Omega} qv\,\mathrm{d}x, \tag{4}$$

where

$$\langle y_3\sigma_1^{\alpha\beta}\rangle = -D_z^{\alpha\beta\lambda\mu}w_{,\lambda\mu}, \tag{5}$$

$$m_\alpha = \{M_\alpha(x,y)(G(y))^{1/2}\}, \quad q = 2\{S(x,y)(G(y))^{1/2}\}. \tag{6}$$

The quantities $\langle\sigma_2^{3\alpha}\rangle$ can be eliminated, which leads to the problem: find $w\in H_0^2(\Omega)$ such that for every $v\in H_0^2(\Omega)$,

$$(\bar{P}_{\text{hom}}^{A}) \qquad tD_z^{\alpha\beta\lambda\mu}\int_{\Omega} w_{,\alpha\beta}v_{,\lambda\mu}\,\mathrm{d}x = \int_{\Omega}(qv-m_\alpha v_{,\alpha})\,\mathrm{d}x. \tag{7}$$

Prior to analyzing the $(P'_{\text{hom}})$ problem, let us examine the local problem $(P_{\text{loc}}^{5})$. The solution to this problem decomposes as $\mathbf{u}^R = \mathbf{u}^S+\mathbf{u}^A$, where $\mathbf{u}^S$ and $\mathbf{u}^A$ are solutions to the following independent problems:

find $\mathbf{u}^S\in W(\mathscr{Y})$ such that for every $\mathbf{w}\in W(\mathscr{Y})$

$$(P_{\text{loc}}^{S}) \qquad a(\mathbf{u}^S,\mathbf{w}) = f^S(\mathbf{w}), \tag{8}$$

where the bilinear form $a(\cdot,\cdot)$ has been defined in Section 5 of Part I and

$$f^S(\mathbf{w}) = \frac{1}{t}\{S_\alpha(x,y)(G(y))^{1/2}[w_\alpha(c,y)+w_\alpha(-c,y)]\}+\langle A_0^{i\beta\gamma\delta}(\mathbf{y})w_i(\mathbf{y})\rangle v_{\gamma,\delta\beta}; \tag{9}$$

find $\mathbf{u}^A\in W(\mathscr{Y})$ such that for every $\mathbf{w}\in W(\mathscr{Y})$

$$(P_{\text{loc}}^{A}) \qquad a(\mathbf{u}^A,\mathbf{w}) = f^A(\mathbf{w}), \tag{10}$$

where

$$f^A(\mathbf{w}) = \frac{1}{t}\left\{\frac{(G(y))^{1/2}}{2\cdot c(y)}\cdot M_\alpha(x,y)\cdot[w_\alpha(c,y)-w_\alpha(-c,y)]\right\}-\langle E_0^{i\beta\gamma\delta}(\mathbf{y})w_i(\mathbf{y})\rangle w_{,\beta\gamma\delta}. \tag{11}$$

Both problems are correctly posed, provided that

$$f^S(\mathbf{w}) = 0, \quad f^A(\mathbf{w}) = 0 \quad \text{for} \quad \mathbf{w} = \text{const}. \tag{12}$$

In virtue of equality (91) and definition $(55)_1$ from Part I and eqn (2), condition $(12)_1$ is fulfilled. Similarly, equality (93) from Part I along with the equation $E_z^{\alpha\beta\lambda\mu} = 0$ yield condition $(12)_2$. The following properties hold true: $u_\alpha^S$, $u_3^A\in S$, $u_\alpha^A$, $u_3^S\in A$ with $x$ being fixed. Note that $u_k^A$ can be expressed as

$$u_k^A = -\Gamma_k^{(\beta\gamma\delta)}(\mathbf{y})w_{,\beta\gamma\delta}+u_k^T(x,y), \tag{13}$$

where $\mathbf{\Gamma}^{(\beta\gamma\delta)}\in W(\mathscr{Y})$ is a solution to the equation

$$(P_{\text{loc}}^{6}) \qquad a(\mathbf{\Gamma}^{(\beta\gamma\delta)},\mathbf{w}) = \langle E_0^{i\beta\gamma\delta}(\mathbf{y})w_i(\mathbf{y})\rangle \tag{14}$$

for every $\mathbf{w}\in W(\mathscr{Y})$, whereas $\mathbf{u}^T\in W(\mathscr{Y})$ fulfils

$$(P^7_{\text{loc}}) \qquad a(\mathbf{u}^T,\mathbf{w}) = \frac{1}{2t}\left\{\frac{[G(y)]^{1/2}}{c(y)} M_\alpha(x,y)[w_\alpha(c,y)-w_\alpha(-c,y)]\right\} \tag{15}$$

for every $\mathbf{w} \in W(\mathscr{Y})$.

In virtue of identity (93) from Part I and the property $E_0^{\gamma\delta\alpha\beta} \in A$, the linear form on the right-hand side of eqn (14) vanishes for $\mathbf{w}$ = const. Similarly, for $\mathbf{w}$ = const, the linear form on the right-hand side of eqn (15) assumes zero values. Hence, we infer that both the local problems are well-posed; their solutions exist and are determined up to additive functions of $x$. Thus, the tensors $p^{\lambda\mu}$ and $m^{\lambda\mu}$ defined by eqns $(56)_{3,4}$ of Part I read

$$\begin{aligned} p^{\lambda\mu} &= \langle C^{\lambda\mu kl} u^S_{k|l}\rangle, \\ m^{\lambda\mu} &= -D_{z3}^{\lambda\mu\beta\gamma\delta} w_{,\beta\gamma\delta} + \langle y_3 C^{\lambda\mu kl} u^T_{k|l}\rangle, \end{aligned} \tag{16}$$

where

$$D_{z3}^{\lambda\mu\beta\gamma\delta} = \langle C^{\lambda\mu kl} y_3 \Gamma_{k|l}^{(\beta\gamma\delta)}\rangle. \tag{17}$$

Equations (64) of Part I assume the form

$$\begin{aligned} \sigma_{20}^{\lambda\mu} &= B_{z2}^{\lambda\mu\gamma\delta\beta} v_{\gamma,\delta\beta} + \langle C^{\lambda\mu kl} u^S_{k|l}\rangle, \\ m_{20}^{\lambda\mu} &= -(D_{z1}^{\lambda\mu\gamma\delta\beta} + D_{z3}^{\lambda\mu\gamma\delta\beta}) w_{,\gamma\delta\beta} + \langle y_3 C^{\lambda\mu kl} u^T_{k|l}\rangle, \end{aligned} \tag{18}$$

which shows that the field $\langle \sigma_2^{\lambda\mu}\rangle$ does not depend upon the field $w$, since it is given by

$$\langle \sigma_2^{\lambda\mu}\rangle = A_z^{\lambda\mu\alpha\beta} z_{\alpha,\beta} + \sigma_{20}^{\lambda\mu}. \tag{19}$$

Similarly, the field $\langle y_3 \sigma_2^{\lambda\mu}\rangle$ is independent of $\mathbf{v} = (v_\alpha)$

$$\langle y_3 \sigma_2^{\lambda\mu}\rangle = -D_z^{\lambda\mu\alpha\beta} v_{3,\alpha\beta} + m_{20}^{\lambda\mu}. \tag{20}$$

Moreover, since $\langle y_3 \Theta_\alpha^{(\gamma\beta)}\rangle = 0$, the function $v_3^0$ which intervenes to $(P'_{\text{hom}})$, is independent of the fields $(v_\alpha)$.

The problem $(P'_{\text{hom}})$ splits up into two independent problems:

the field $\mathbf{z} = (z_\alpha) \in [H^1_0(\Omega)]^2$ fulfils

$$(P'^S_{\text{hom}}) \qquad \int_\Omega A_z^{\alpha\beta\lambda\mu} z_{\alpha,\beta} w_{\lambda,\mu}\,\mathrm{d}x = -\int_\Omega \sigma_{20}^{\lambda\mu} w_{\lambda,\mu}\,\mathrm{d}x \tag{21}$$

for every $\mathbf{w} \in [H^1_0(\Omega)]^2$;

the field $v_3$ is such that $v_3 - v_3^0 \in H^2_0(\Omega)$ and

$$(P'^A_{\text{hom}}) \qquad \int_\Omega D_z^{\alpha\beta\lambda\mu} v_{3,\alpha\beta} v_{,\lambda\mu}\,\mathrm{d}x = -\int_\Omega m_{20}^{\lambda\mu} \cdot v_{,\lambda\mu}\,\mathrm{d}x \tag{22}$$

for every $v \in H^2_0(\Omega)$.

It has been shown that in the case considered (symmetry with respect to the $x_3 = 0$ plane), the displacement and stress fields can be divided into fields that contribute to the membrane problem and fields that contribute only to the bending problem. These former fields depend upon the fields $(v_\alpha)$ and $(z_\alpha)$ which are solutions to the $(P^S_{\text{hom}})$ and $(P'^S_{\text{hom}})$ problems, while the latter fields, determined by $w$ and $v_3$, are solutions to the bending problems $(\bar{P}^A_{\text{hom}})$ and $(P'^A_{\text{hom}})$. Consider the case when a plate is transversely loaded, viz.

$r_\alpha^\pm = 0$. Then, $v_\alpha = 0$ and consequently $\mathbf{u}^S = \mathbf{0}$, $\mathbf{u}^T = \mathbf{0}$. Thus, the fields $(\sigma_{20}^{\lambda\mu})$ vanish, and hence $z_\alpha = 0$.

The formulae for displacements (39)–(42) in Part I reduce to the form

$$\begin{aligned}
u_\alpha^{(0)} &= 0, \quad u_3^{(0)} = w,\\
u_\alpha^{(1)} &= -y_3 w_{,\alpha}, \quad u_3^{(1)} = v_3,\\
u_\alpha^{(2)} &= -\Xi_\alpha^{(\gamma\beta)} w_{,\gamma\beta} - y_3 v_{3,\alpha},\\
u_3^{(2)} &= -\Xi_3^{(\gamma\beta)} w_{,\gamma\beta} + z_3,\\
u_\alpha^{(3)} &= -\Xi_\alpha^{(\gamma\delta)} v_{3,\gamma\delta} - (\Pi_\alpha^{(\gamma\delta\beta)} + \Gamma_\alpha^{(\gamma\delta\beta)}) w_{,\gamma\delta\beta} + \bar{z}_\alpha - y_3 z_{3,\alpha},\\
u_3^{(3)} &= -\Xi_3^{(\gamma\delta)} v_{3,\gamma\delta} - (\Pi_3^{(\gamma\delta\beta)} + \Gamma_3^{(\gamma\delta\beta)}) w_{,\gamma\delta\beta} + \bar{z}_3\,;
\end{aligned} \tag{23}$$

whereas the stresses are given by

$$\begin{aligned}
\sigma_0^{ij} &= 0,\\
\sigma_1^{ij} &= -E_0^{ij\alpha\beta} w_{,\alpha\beta},\\
\sigma_2^{ij} &= -E_0^{ij\alpha\beta} v_{3,\alpha\beta} - [B_1^{ij\gamma\delta\beta} + C^{ijkl}\Gamma_{k|l}^{(\beta\gamma\delta)}]\cdot w_{,\gamma\delta\beta}.
\end{aligned} \tag{24}$$

Note that the stress fields (24) are independent of the extra fields $z_3$ and $\bar{z}_3$, which cannot be determined from the $(\bar{P}^A_{\text{hom}})$ and $(P'^A_{\text{hom}})$ problems.

## 3. FORMULAE FOR PLATES, THE PERIODICITY CELLS OF WHICH POSSESS SYMMETRY PROPERTIES WITH RESPECT TO THE PLANES $y_i = 0$

As in the previous section, we will consider plates symmetric with respect to the $x_3 = 0$-plane. The planes $x_3 =$ const. are still planes of material symmetry. Moreover, we assume that the planes $x_\alpha =$ const are also planes of material symmetry. Thus we confine our consideration to plates made of a material which is orthotropic with respect to the coordinate system $(x_i)$. Hence, the only non-zero components of the elasticity tensors are $C^{iijj} = C^{jjii}$ and $C^{ijij} = C^{jiji} = C^{ijji}$. Let us shift the local coordinate system $(y_i)$ to the centre of the cell, cf. Fig. 1. We shall additionally assume that the planes $y_\alpha = 0$ are planes of symmetry of the cell $\mathcal{Y}$, i.e.

$$\begin{aligned}
C^{ijkl}(y_3, y_1, y_2) &= C^{ijkl}(\pm y_3, \pm y_1, \pm y_2),\\
G(y_1, y_2) = G(\pm y_1, \pm y_2), &\quad c(y_1, y_2) = c(\pm y_1, \pm y_2).
\end{aligned} \tag{25}$$

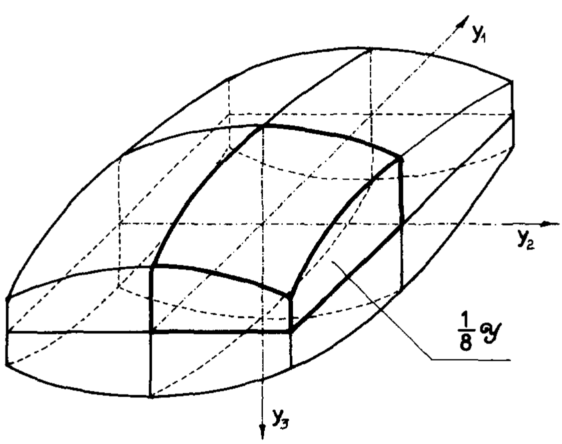


Fig. 1. Geometry of the rescaled cell of periodicity.

Similarly, the loadings are assumed to vary symmetrically with respect to the micro-coordinates, i.e.

$$r_\alpha^\pm(x;y_1,y_2) = r_\alpha^\pm(x;\pm y_1,\pm y_2),\quad q^\pm(x;y_1,y_2) = q^\pm(x;\pm y_1,\pm y_2). \tag{26}$$

Thus the cell $\mathscr{Y}$ is composed of eight identical segments, cf. Fig. 1. In the following, the consequences of the assumed symmetry properties will be examined.

To shorten any further formulae, we now introduce denotations of classes of functions defined on $\mathscr{Y}$ which are odd or even with respect to $y_3$, $y_1$ and $y_2$. We write $f \in SSA$, if $f$ defined on $\mathscr{Y}$ is such that $f(y_3, y_1, y_2)$ is even with respect to $y_3$ and $y_1$ and odd with respect to $y_2$. Similarly, we define the classes $SSS$, $AAA$, $SAA$, etc. For the functions defined on $Y$ we use the classes $SS$, $AS$, $SA$ and $AA$. For instance, conditions (25) imply $C^{ijkl} \in SSS$, $G \in SS$, $c \in SS$.

Let us start from the problem $(P^7_{\text{loc}})$. Since $M_\alpha(x,\cdot) \in SS$, one can show (we omit the proof) that

$$\langle y_3 C^{\lambda\mu kl} u^T_{k|l} \rangle = 0 \tag{27}$$

and thus the last term in formula $(18)_2$ for $m^{\lambda\mu}_{20}$ vanishes.

Consider the $(P^2_{\text{loc}})$ problem in Section 5 of Part I. Taking into account the orthotropy of the material along with the properties of symmetry (25), one can verify that

$$\Xi_1^{(\alpha\alpha)} \in AAS,\quad \Xi_2^{(\alpha\alpha)} \in ASA,\quad \Xi_3^{(\alpha\alpha)} \in SSS, \tag{28}$$

$$\Xi_1^{(12)} \in ASA,\quad \Xi_2^{(12)} \in AAS,\quad \Xi_3^{(12)} \in SAA. \tag{29}$$

Having found the above properties, one can establish the properties of the solution to the $(P^4_{\text{loc}})$ problem. Its solution can be decomposed into a sum:

$$\boldsymbol{\Pi}^{(\beta\gamma\delta)} = \boldsymbol{\Pi}_3^{(\beta\gamma\delta)} + \boldsymbol{\Pi}_1^{(\beta\gamma\delta)} + \boldsymbol{\Pi}_2^{(\beta\gamma\delta)}, \tag{30}$$

where $\boldsymbol{\Pi}_l^{(\beta\gamma\delta)} \in W(\mathscr{Y})$ is a solution to the variational equation

$$(P^{4l}_{\text{loc}}) \qquad a(\boldsymbol{\Pi}_l^{(\beta\gamma\delta)}, \mathbf{w}) + \langle C^{ijl\beta} \Xi_l^{(\gamma\delta)} w_{i|j} \rangle = 0 \tag{31}$$

for every $\mathbf{w} \in W(\mathscr{Y})$ (do not sum over $l$).

One can now examine the symmetry properties of the components $\Pi^{(\beta\gamma\delta)}_{lk}$. For instance, $\Pi^{(212)}_{31} \in ASS$, etc. In virtue of these properties and bearing in mind relations (29), one can verify that all the components of the tensor $\mathbf{D}_{z1}$ defined by eqns (56) and (54) of Part I are zero.

Let us focus our attention on the $(P^6_{\text{loc}})$ problem. It should be decomposed into three problems:

find $\boldsymbol{\Gamma}_l^{(\beta\gamma\delta)} \in W(\mathscr{Y})$ such that for every $\mathbf{w} \in W(\mathscr{Y})$

$$(P^{6l}_{\text{loc}}) \qquad a(\boldsymbol{\Gamma}_l^{(\beta\gamma\delta)}, \mathbf{w}) = \langle E_0^{l\beta\gamma\delta}(\mathbf{y}) w_l(\mathbf{y}) \rangle \quad \text{(do not sum over } l). \tag{32}$$

After this decomposition, one can determine the symmetry properties of the components $\Gamma^{(\beta\gamma\delta)}_{lk}$. At first one should find the symmetry properties of $E_0^{l\beta\gamma\delta}$ (e.g. $E_0^{2212} \in AAA$, etc.) which are "body forces". Knowing these properties, one can determine the symmetry properties of the fields $\Gamma^{(\beta\gamma\delta)}_{lk}$, e.g. $\Gamma^{(212)}_{12} \in AAA$, etc., and check that the tensor $\mathbf{D}_{z3}$ defined by (17) vanishes. Eventually, we arrive at the conclusion that $m^{\lambda\mu}_{20} = 0$.

Let us come back to the $(P^1_{\text{loc}})$ problem [for $(kl) = (\alpha\beta)$] whose solutions possess the following properties:

$$\Theta_1^{(\alpha\alpha)} \in SAS, \quad \Theta_2^{(\alpha\alpha)} \in SSA, \quad \Theta_3^{(\alpha\alpha)} \in ASS,$$
$$\Theta_1^{(12)} \in SSA, \quad \Theta_2^{(12)} \in SAS, \quad \Theta_3^{(12)} \in AAA, \tag{33}$$

which makes it possible to find symmetry classes of the fields $A_0^{3\beta\gamma\delta}$ and $A_0^{\alpha\beta\lambda\mu}$, e.g. $A_0^{\alpha\alpha\beta\beta} \in SSS$.

The solution to the problem $(P_{\text{loc}}^S)$ can be written down as

$$u_k^S = Z_k^{(\beta\gamma\delta)}(\mathbf{y}) \cdot v_{\gamma,\beta\delta} + u_k^{S2}(x, \mathbf{y}). \tag{34}$$

The functions $\mathbf{Z}^{(\beta\gamma\delta)} \in W(\mathscr{Y})$ fulfil the following equation

$$(P_{\text{loc}}^{S1}) \qquad a(\mathbf{Z}^{(\beta\gamma\delta)}, \mathbf{w}) = \langle A_0^{3\beta\gamma\delta}(\mathbf{y}) \cdot w_3(\mathbf{y}) \rangle \tag{35}$$

for every $\mathbf{w} \in W(\mathscr{Y})$.

The function $\mathbf{u}^{S2}(x, \cdot) \in W(\mathscr{Y})$ satisfies

$$(P_{\text{loc}}^{S2}) \qquad a(\mathbf{u}^{S2}, \mathbf{w}) = f^{S2}(\mathbf{w}) \quad \text{for every } \mathbf{w} \in W(\mathscr{Y}), \tag{36}$$

where

$$f^{S2}(\mathbf{w}) = \langle A_0^{\alpha\beta\gamma\delta}(\mathbf{y}) w_\alpha(\mathbf{y}) \rangle v_{\gamma,\delta\beta} + \frac{1}{t}\{S_\alpha(x, y) \cdot [G(y)]^{1/2}[w_\alpha(c, y) + w_\alpha(-c, y)]\}. \tag{37}$$

By virtue of formula (91) of Part I, the $(P_{\text{loc}}^{S1})$ problem is correctly posed. Similarly, eqn (2) assures that the $(P_{\text{loc}}^{S2})$ problem is well stated. We omit the rather lengthy proof that

$$\langle C^{\lambda\mu kl} Z_{k|l}^{(\beta\gamma\delta)} \rangle = 0, \quad \langle C^{\lambda\mu kl} u_{k|l}^{S2} \rangle = 0 \tag{38}$$

and hence the last term in the definition of $\sigma_{20}^{\lambda\mu}$ [eqn (18)] is zero.

Lastly, let us examine whether the tensor

$$B_{z2}^{\lambda\mu\gamma\delta\beta} = \langle C^{\lambda\mu\sigma\beta} \Theta_\sigma^{(\gamma\delta)} + C^{\lambda\mu kl} \Lambda_{k|l}^{(\beta\gamma\delta)} \rangle, \tag{39}$$

defined by eqn (56), of Part I vanishes. Relations (33) imply that the first term in its definition is equal to zero. Prior to analyzing the second term, let us examine the solutions of the $(P_{\text{loc}}^3)$ problem. This problem should be decomposed into three problems:

find $\Lambda_l^{(\sigma\gamma\beta)} \in W(\mathscr{Y})$ such that

$$(P_{\text{loc}}^{3l}) \qquad a(\Lambda_l^{(\sigma\gamma\beta)}, \mathbf{w}) + \langle C^{ijl\sigma} \Theta_l^{(\gamma\beta)} w_{i|j} \rangle = 0 \tag{40}$$

for every $\mathbf{w} \in W(\mathscr{Y})$; do not sum over $l$.

From the properties in (33) which have been found, the properties of $\Lambda_k^{(\sigma\gamma\beta)}$ may subsequently be determined. Upon establishing these properties, one can conclude that the second term in (39) vanishes, which corroborates that the tensor $\mathbf{B}_{z2}$ is zero. Thus, we have eventually proved that in the case considered,

$$\sigma_{20}^{\lambda\mu} = 0 \quad \text{and} \quad m_{20}^{\lambda\mu} = 0 \quad \text{hold.} \tag{41}$$

Moreover, the right-hand side of eqn (104) of Part I vanishes by virtue of properties (33) and (29). Consequently, $v_3^0 = 0$. Note that in the case considered, eqn (104) from Part I does not result in any discrepancy between the values of $v_3$ and $\partial v_3/\partial s$ along $\gamma$. Thus, both problems $(P_{\text{hom}}^{\prime S})$ and $(P_{\text{hom}}^{\prime A})$ are homogeneous and their solutions are trivial:

$$z_\alpha = 0, \quad v_3 = 0. \tag{42}$$

The solution (39)–(42) and (43)–(45) of the $(P_\varepsilon)$ problem from Part I can be reduced as follows. The displacements read

$$\begin{aligned} u^\varepsilon_\alpha &= \varepsilon(\underline{v_\alpha} - y_3 w_{,\alpha}) + \varepsilon^2 \cdot [\underline{\Theta^{(\gamma\beta)}_\alpha(\mathbf{y}) v_{\gamma,\beta}} - \Xi^{(\gamma\beta)}_\alpha(\mathbf{y}) w_{,\gamma\beta}] + \mathrm{O}(\varepsilon^3), \\ u^\varepsilon_3 &= w + \varepsilon^2[\underline{\Theta^{(\gamma\beta)}_3(\mathbf{y}) v_{\gamma,\beta}} - \Xi^{(\gamma\beta)}_3(\mathbf{y}) w_{,\gamma\beta} + z_3] + \mathrm{O}(\varepsilon^3), \end{aligned} \tag{43}$$

where one should insert $\mathbf{y} = \mathbf{x}/\varepsilon$.

The stresses are given by

$$\begin{gathered} \sigma^{ij}_\varepsilon = \varepsilon[\underline{A^{ij\alpha\beta}_0(\mathbf{y}) v_{\beta,\alpha}} - E^{ij\alpha\beta}_0(\mathbf{y}) w_{,\alpha\beta}] + \varepsilon^2 \sigma^{ij}_2 + \mathrm{O}(\varepsilon^3) \\ \sigma^{ij}_2 = -B^{ij\gamma\delta\beta}_1(\mathbf{y}) w_{,\gamma\delta\beta} + \underline{B^{ij\gamma\delta\beta}_2(\mathbf{y}) v_{\gamma,\delta\beta}} + C^{ijkl}(\mathbf{y})[\underline{u^{S2}_{k|l}} - \Gamma^{(\beta\gamma\delta)}_{k|l}(\mathbf{y}) w_{,\beta\gamma\delta} + \underline{u^T_{k|l} + Z^{(\beta\gamma\delta)}_{k|l}(\mathbf{y}) v_{\gamma,\beta\delta}}], \end{gathered} \tag{44}$$

where $\mathbf{y} = \mathbf{x}/\varepsilon$.

The field $z_3$ can be determined in the subsequent step of the asymptotic process. The fields $v_\alpha$ and $w$ are solutions to the independent problems $(P^S_{\text{hom}})$ and $(P^A_{\text{hom}})$. In the case of bending, the underlined terms vanish.

Let us emphasize once more that the equality $z_\alpha = 0$ is an approximation that is a consequence of averaging the boundary condition (103) from Part I over $\mathcal{Y}$.

## 4. ASYMPTOTIC EXPANSIONS AS KINEMATICAL HYPOTHESES

The aim of this section is to put forward a direct derivation of the $(P_{\text{hom}})$ problem. The "ansatz" (31) in Part I, will be used as constraints to be imposed on the variational equation (30) in Part I. Let the unknown and test displacement functions of eqn (30), Part I be put in the form

$$\begin{aligned} \mathbf{U}^\varepsilon &= \mathbf{u}^{(0)}(x) + \varepsilon \mathbf{u}^{(1)}(x, \mathbf{y}) + \varepsilon^2 \cdot \mathbf{u}^{(2)}(x, \mathbf{y}), \\ \mathbf{V} &= \mathbf{v}^{(0)}(x) + \varepsilon \mathbf{v}^{(1)}(x, \mathbf{y}) + \varepsilon^2 \cdot \mathbf{v}^{(2)}(x, \mathbf{y}). \end{aligned} \tag{45}$$

The fields $\mathbf{u}^{(p)}$ are given by eqns (39)–(41) from Part I, the fields $v^{(p)}$ being defined by similar formulae in which the fields $\tilde{w}$, $\tilde{v}_k$ and $\tilde{z}_k$ are substituted for the fields $w$, $v_k$ and $z_k$. The deformations compatible with displacements (45) are

$$\begin{aligned} \gamma_{ij}(\mathbf{U}^\varepsilon) &= a^{(\alpha\beta)}_{ij}(\mathbf{y}) \cdot \gamma_{\alpha\beta}(\mathbf{w}) + \varepsilon e^{(\alpha\beta)}_{ij}(\mathbf{y}) \cdot \kappa_{\alpha\beta}(w) + \mathrm{O}(\varepsilon^2), \\ \gamma_{kl}(\mathbf{V}) &= a^{(\alpha\beta)}_{kl}(\mathbf{y}) \cdot \gamma_{\alpha\beta}(\tilde{\mathbf{w}}) + \varepsilon e^{(\alpha\beta)}_{kl}(\mathbf{y}) \cdot \kappa_{\alpha\beta}(\tilde{w}) + \mathrm{O}(\varepsilon^2), \end{aligned} \tag{46}$$

where $\mathbf{w} = (w_\alpha)$, $\tilde{\mathbf{w}} = (\tilde{w}_\alpha)$, $w_\alpha = \varepsilon v_\alpha$, $\tilde{w}_\alpha = \varepsilon \tilde{v}_\alpha$; and the strain measures are defined by

$$\gamma_{\alpha\beta}(\mathbf{w}) = \tfrac{1}{2}(w_{\alpha,\beta} + w_{\beta,\alpha}), \quad \kappa_{\alpha\beta}(w) = -w_{,\alpha\beta}. \tag{47}$$

The fields $a^{(\alpha\beta)}_{ij}$ and $e^{(\alpha\beta)}_{ij}$ are defined by (51) in Part I. The bilinear form (27) in Part I reduces to

$$A^\varepsilon(\mathbf{U}^\varepsilon, \mathbf{V}) = a(\mathbf{w}, w; \tilde{\mathbf{w}}, \tilde{w}) + \mathrm{O}(\varepsilon^5) \tag{48}$$

where

$$a(\mathbf{w}, w; \tilde{\mathbf{w}}, \tilde{w}) = \int_\Omega [N_h^{\alpha\beta}(\mathbf{w}, w)\gamma_{\alpha\beta}(\tilde{\mathbf{w}}) + M_h^{\alpha\beta}(\mathbf{w}, w)\kappa_{\alpha\beta}(\tilde{w})]\,\mathrm{d}x. \tag{49}$$

The above bilinear form depends on $\varepsilon$ but the index $\varepsilon$ is omitted, which should not result in any misunderstanding.

The overall constitutive relationships read

$$\begin{aligned} N_h^{\alpha\beta} &= A_{\mathrm{hom}}^{\alpha\beta\lambda\mu}\gamma_{\lambda\mu}(\mathbf{w}) + E_{\mathrm{hom}}^{\alpha\beta\lambda\mu}\kappa_{\lambda\mu}(w), \\ M_h^{\alpha\beta} &= F_{\mathrm{hom}}^{\alpha\beta\lambda\mu}\gamma_{\lambda\mu}(\mathbf{w}) + D_{\mathrm{hom}}^{\alpha\beta\lambda\mu}\kappa_{\lambda\mu}(w), \end{aligned} \tag{50}$$

where the stiffnesses are given by

$$(\mathbf{A}_{\mathrm{hom}}, \mathbf{E}_{\mathrm{hom}}, \mathbf{F}_{\mathrm{hom}}, \mathbf{D}_{\mathrm{hom}}) = t\cdot(\varepsilon\mathbf{A}_z, \varepsilon^2\mathbf{E}_z, \varepsilon^2\mathbf{F}_z, \varepsilon^3\mathbf{D}_z). \tag{51}$$

The above relations have been derived with the help of formulae (77) and (85) of Part I. The stress resultants (50) are related to the field $\sigma_1^{\alpha\beta}$, namely

$$N_h^{\alpha\beta} = \varepsilon^2\langle\sigma_1^{\alpha\beta}\rangle, \quad M_h^{\alpha\beta} = \varepsilon^3\langle\hat{y}_3\sigma_1^{\alpha\beta}\rangle. \tag{52}$$

Upon discarding the terms of higher order with respect to $\varepsilon$, one can reduce the linear form $F^\varepsilon$, defined by (26) in Part I, to

$$F^\varepsilon(\mathbf{V}) = f(\tilde{\mathbf{w}}, \tilde{w}) = \int_\Omega (q^*\tilde{w} - m_\alpha^*\tilde{w}_{,\alpha} + r_\alpha^*\tilde{w}_\alpha)\,\mathrm{d}x, \tag{53}$$

where

$$q^* = \varepsilon^3\cdot q, \quad r_\alpha^* = \varepsilon^2 r_\alpha, \quad m_\alpha^* = \varepsilon^3 m_\alpha. \tag{54}$$

The loading functions (54) are related to the given densities of loads [cf. (6), Part I] according to the formulae below

$$\begin{aligned} q^* &= \{p_3^-(x,y)[G_-(y)]^{1/2}\} + \{p_3^+(x,y)[G_+(y)]^{1/2}\}, \\ r_\alpha^* &= \{p_\alpha^-(x,y)[G_-(y)]^{1/2}\} + \{p_\alpha^+(x,y)[G_+(y)]^{1/2}\}, \\ m_\alpha^* &= \{p_\alpha^-(x,y)\hat{h}^-(y)(G_-(y))^{1/2}\} + \{p_\alpha^+(x,y)\hat{h}^+(y)(G_+(y))^{1/2}\}, \end{aligned} \tag{55}$$

where $\hat{h}^\pm = h^\pm - \varepsilon\langle y_3\rangle$.

The $(P_{\mathrm{hom}})$ problem formulated in Section 5 of Part I is equivalent to the following problem:

find $(\mathbf{w}, w) \in [H_0^1(\Omega)]^2 \times H_0^2(\Omega)$ such that

$$(\bar{P}_{\mathrm{hom}}) \qquad a(\mathbf{w}, w; \tilde{\mathbf{w}}, \tilde{w}) = f(\tilde{\mathbf{w}}, \tilde{w}) \quad \text{for every } (\tilde{\mathbf{w}}, w) \in [H_0^1(\Omega)]^2 \times H_0^2(\Omega). \tag{56}$$

Note that for $\varepsilon = \varepsilon_0$, the above problem is an effective problem for the original problem $(P)$, cf. Section 2, Part I. Consequently, the stiffnesses (51) for $\varepsilon = \varepsilon_0$ characterize the original plate.

## 5. THE KOHN AND VOGELIUS APPROACH

As in the papers by Kohn and Vogelius (1984, 1985), let us focus our attention on plates of periodically varying thickness, symmetric with respect to the mid-plane, made from an elastic homogeneous material and subjected to transverse loadings. Thus only the bending response will be dealt with. The interrelations between formulae (23) and (24) and the formulae discussed in Kohn and Vogelius (1984), for $a = 1$, will be disclosed below. Let us recall eqns (4.5) from Kohn and Vogelius (1984)

$$\begin{aligned} u_\alpha^\varepsilon &= -\varepsilon y_3 w_{,\alpha} + \varepsilon^2 \Phi_\alpha^{(\lambda\mu)}(\mathbf{y}) w_{,\lambda\mu}, \\ u_3^\varepsilon &= w + \frac{1}{2}\frac{C^{\alpha\beta 33}}{C^{3333}}\varepsilon^2 (y_3)^2 w_{,\alpha\beta} + \varepsilon^2 \Phi_3^{(\lambda\mu)}(\mathbf{y}) w_{,\lambda\mu}. \end{aligned} \tag{57}$$

The functions $\mathbf{\Phi}^{(\lambda\mu)} \in W(\mathscr{Y})$ satisfy

$$(P_{\text{loc}}^{KV}) \qquad C^{ijkl} \cdot \Phi_{k|lj}^{(\lambda\mu)} = 0 \quad \text{in} \quad \mathscr{Y} \tag{58}$$

and

$$\begin{aligned} C^{\alpha jkl}\Phi_{k|l}^{(\lambda\mu)} n_j^{\pm} &= \pm c(y)\tilde{C}^{\alpha\gamma\lambda\mu} n_\gamma^{\pm} \\ C^{3jkl}\Phi_{k|l}^{(\lambda\mu)} n_j^{\pm} &= 0 \quad \text{on the faces of } \mathscr{Y}. \end{aligned} \tag{59}$$

The tensor $\tilde{C}^{ij\alpha\beta}$ is defined by

$$\tilde{C}^{ij\alpha\beta} = C^{ij\alpha\beta} - (C^{3333})^{-1} \cdot C^{\alpha\beta 33} \cdot C^{ij33}. \tag{60}$$

In particular,

$$\tilde{C}^{33\alpha\beta} = \tilde{C}^{3\lambda\alpha\beta} = 0. \tag{61}$$

The tensor $\tilde{C}^{\alpha\beta\lambda\mu}$ stands for the tensor of elastic moduli of the generalized plane-stress problem, $\sigma^{33} \approx 0$. The functions $\mathbf{\Xi}^{(\lambda\mu)}$ and $\mathbf{\Phi}^{(\lambda\mu)}$ are interrelated according to the formulae

$$\begin{aligned} \Xi_\alpha^{(\lambda\mu)} &= -\Phi_\alpha^{(\lambda\mu)} + f_\alpha^{(\lambda\mu)}, \\ \Xi_3^{(\lambda\mu)} &= -\Phi_3^{(\lambda\mu)} - \tfrac{1}{2}(y_3)^2 \cdot \frac{C^{\lambda\mu 33}}{C^{3333}} + f_3^{(\lambda\mu)}, \end{aligned} \tag{62}$$

where $f_i^{(\lambda\mu)}$ do not depend on $\mathbf{y}$.

To prove the above relations, let us express the fields $E_0^{ij\alpha\beta}$ defined by $(50)_2$ of Part I in terms of the functions $\Phi_i^{(\lambda\mu)}$:

$$E_0^{ij\alpha\beta} = y_3\tilde{C}^{ij\alpha\beta} - C^{ijkl}\Phi_{k|l}^{(\alpha\beta)}. \tag{63}$$

Hence, eqn (47) in Part I assumes the form

$$a(\mathbf{\Phi}^{(\alpha\beta)}, \mathbf{w}) = \tilde{C}^{ij\alpha\beta}\langle y_3 w_{i|j}\rangle. \tag{64}$$

The right-hand side of eqn (64) can be rearranged as

$$\tilde{C}^{ij\alpha\beta}\langle (y_3 w_i)_{|j}\rangle - \tilde{C}^{i3\alpha\beta}\langle w_i\rangle. \tag{65}$$

In virtue of (61), the second term vanishes. Changing the volume integral into a surface integral and bearing in mind that the $w_i$ are $Y$-periodic in $y$, we reduce the first term of (65) to

$$\frac{1}{|\mathscr{Y}|}\cdot \tilde{C}^{ij\alpha\beta}\cdot \int_{Y} [n_j^+(y)\cdot c(y)w_i(c(y),y)-n_j^-(y)c(y)w_i(-c(y),y)](G(y))^{1/2}\,\mathrm{d}y. \tag{66}$$

The standard arguments of variational calculus now leads us to the formulation $(P_{\text{loc}}^{KV})$. Thus, taking apart the subtle regularity conditions in the special case considered, both the local problems $(P_{\text{loc}}^{2})$ and $(P_{\text{loc}}^{KV})$ are mutually equivalent. However, formulae (57) and (23) do not coincide; they differ in terms involving the fields $v_3$ and $z_3$. These fields do not affect the stress field $\sigma_1^{ij}$ and that is why they have been discarded in the Kohn and Vogelius analysis. In the trivial case where $c = \text{const}$, the functions $\mathbf{\Phi}^{(\alpha\beta)}$ vanish. Hypotheses (57) reduce to the Nordgren-type kinematical assumptions (Nordgren, 1971; cf. also Lewiński, 1987, where a brief survey of kinematical assumptions has been given).

## 6. THE GALERKIN ALGORITHM FOR COMPUTING EFFECTIVE STIFFNESS

This section is concerned with a composite plate as described in Section 2 of Part I (i.e. the additional assumptions stipulated in Section 2 need not be fulfilled here). Our aim is to give an algorithm for computing the rigidities (55), Part I, and hence the rigidities (51) of the initial plate.

The auxiliary functions $\mathbf{\Theta}^{(\alpha\beta)}$ and $\mathbf{\Xi}^{(\alpha\beta)}$, which enter the definitions (55) from Part I of effective stiffnesses, can be approximated by the Galerkin method. Let $(h_a)_{a=1}^{N}$ be a basis in $N$-dimensional sub-space $W_N(\mathscr{Y})$ of the space $W(\mathscr{Y})$. Additionally, we impose the condition

$$\langle h_a\rangle = 0, \quad a = 1,\ldots,N. \tag{67}$$

The unknown auxiliary functions are approximated by functions from the space $W_N(\mathscr{Y})$, and hence we decompose them in the basis $(h_a)$

$$\Theta_k^{(\alpha\beta)} \approx \Theta_k^{(\alpha\beta)a}h_a(\mathbf{y}), \quad \Xi_k^{(\alpha\beta)} \approx \Xi_k^{(\alpha\beta)a}h_a(\mathbf{y}). \tag{68}$$

Similarly, the test functions $\mathbf{w}$ are taken from $W_N(\mathscr{Y})$. The coefficients $\Theta_k^{(\alpha\beta)a}$ and $\Xi_k^{(\alpha\beta)a}$ are solutions to the following algebraic equations

$$C_{ba}^{ik}\Theta_k^{(\alpha\beta)a}+C_b^{i(\alpha\beta)} = 0, \quad C_{ba}^{ik}\Xi_k^{(\alpha\beta)a}+E_b^{i(\alpha\beta)} = 0, \tag{69}$$

where

$$C_{ba}^{ik} = \langle C^{ijkl}h_{a|l}h_{b|j}\rangle,$$
$$C_b^{i(\alpha\beta)} = \langle C^{ij\alpha\beta}h_{b|j}\rangle \quad \text{and} \quad E_b^{i(\alpha\beta)} = \langle \hat{y}_3 C^{ij\alpha\beta}h_{b|j}\rangle. \tag{70}$$

Let the matrix $[B_{ki}^{ab}]$ represent the inverse of the matrix $[C_{ba}^{ik}]$. The solutions to eqns (69) can be written down as

$$\Theta_k^{(\alpha\beta)a} = -B_{ki}^{ab}C_b^{i(\alpha\beta)}, \quad \Xi_k^{(\alpha\beta)a} = -B_{ki}^{ab}E_b^{i(\alpha\beta)}. \tag{71}$$

The tensors $\mathbf{A}_0$ and $\mathbf{E}_0$, which determine the leading term $\sigma_1^{ij}$ in the sequence (32) of Part I [cf. eqns (43)–(44) of Part I], are expressed as

$$A_0^{\gamma\delta\alpha\beta} = -C_b^{i(\alpha\beta)}B_{ki}^{ab}\cdot C^{\gamma\delta kl}h_{a|l}+C^{\gamma\delta\alpha\beta},$$
$$E_0^{\gamma\delta\alpha\beta} = -E_b^{i(\alpha\beta)}B_{ki}^{ab}\cdot C^{\gamma\delta kl}h_{a|l}+\hat{y}_3C^{\gamma\delta\alpha\beta}. \tag{72}$$

The effective stiffnesses can be computed by the formulae

$$
\begin{aligned}
A_z^{\gamma\delta\alpha\beta} &= \langle C^{\gamma\delta\alpha\beta}\rangle - C_a^{k(\gamma\delta)} B_{ki}^{ab} C_b^{i(\alpha\beta)}, \\
E_z^{\gamma\delta\alpha\beta} &= \langle \hat{y}_3 C^{\gamma\delta\alpha\beta}\rangle - E_b^{i(\alpha\beta)} B_{ki}^{ab} C_a^{k(\gamma\delta)}, \\
F_z^{\gamma\delta\alpha\beta} &= \langle \hat{y}_3 C^{\gamma\delta\alpha\beta}\rangle - E_a^{k(\gamma\delta)} B_{ki}^{ab} C_b^{i(\alpha\beta)}, \\
D_z^{\gamma\delta\alpha\beta} &= \langle (\hat{y}_3)^2 C^{\gamma\delta\alpha\beta}\rangle - E_a^{k(\gamma\delta)} B_{ki}^{ab} E_b^{i(\alpha\beta)}.
\end{aligned} \tag{73}
$$

The main difficulty in implementing the above algorithm lies in forming the basis $(h_a)$, which satisfies condition (67). The finite element bases that fulfil this condition have been constructed by Żmijewski (1987).

## 7. CONCLUDING REMARKS

Consideration of Section 2 confirms that if a plate is symmetric with respect to its mid-plane, the membrane and bending effects become independent. This decoupling concerns both the global and local problems that constitute the subsequent terms of the asymptotic solution. Less obvious simplifications follow from the symmetries of the plate material and of the periodicity cell with respect to its central transverse sections. In particular, the solutions to the subsidiary global problem $(P'_{\text{hom}})$ are zero. Moreover, the boundary condition (104) in Part I can be identically satisfied and no contradications between the values of $v_3$ and $\partial v_3/\partial s$ along the boundary are observed.

According to the results of Section 5, the formulae for bending stiffnesses from Kohn and Vogelius (1984) can be obtained from those derived in Part I. However, it has been disclosed that the kinematical "ansatz" used by Kohn and Vogelius (1984) does not include some terms which can be important in evaluating displacements.

## REFERENCES


Bensoussan, A., Lions, J.-L. and Papanicolaou, G. (1978). *Asymptotic Analysis for Periodic Structures*. North-Holland, Amsterdam.

Bourgat, J.-F. and Dervieux, A. (1978). Méthode d'homogénéisation des opérateurs à coefficients périodiques: étude des correcteurs provenant du développement asymptotique. IRIA Rapport de Recherche No. 278, Rocquencourt.

Caillerie, D. (1984). Thin elastic and periodic plates. *Math. Meth. Appl. Sci.* **6,** 159–191.

Kalamkarov, A. L., Kudryavtsev, B. A. and Parton, V. Z. (1987). A problem of a curved composite layer with wavy faces of periodic geometry (in Russian). *Prikl. Mat. Mekh.* **51,** 68–75.

Kohn, R. V. and Vogelius, M. (1984). A new model for thin plates with rapidly varying thickness. *Int. J. Solids Structures* **20,** 333–350.

Kohn, R. V. and Vogelius, M. (1985). A new model for thin plates with rapidly varying thickness II: a convergence proof. *Q. Appl. Math.* **43,** 1–22.

Lewiński, T. (1987). On refined plate models based on kinematical assumptions. *Ing.-Arch.* **57,** 133–146.

Lewiński, T. (1991). Effective models of composite periodic plates—I. Asymptotic solution. *Int. J. Solids Structures* **27,** 1155–1172.

Nordgren, R. P. (1971). A bound on the error in plate theory. *Q. Appl. Math.* **28,** 587–595.

Żmijewski, K. H. (1987). Numerical implementation of the microlocal parameters method (in Polish). In *Proc. Conf. on Computer Methods in Structural Mechanics*, Jadwisin near Warsaw, May 1987 (Edited by M. Kleiber), pp. 471–480. Politechnika Warszawska Press.